

AN ISOMORPHISM THEOREM FOR ANTI-ORDERED SETS

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Abstract

In this paper we show some kind of isomorphism theorem for ordered sets under antiorders. Let $(X, =_X, \neq_X, \alpha)$ and $(Y, =_Y, \neq_Y, \beta)$ be ordered sets under antiorders, where the apartness $\neq_Y$ is tight. If $\varphi : X \rightarrow Y$ is reverse isotone strongly extensional mapping, then there exists a strongly extensional and embedding reverse isotone bijection

$$((X, =_X, \neq_X, \alpha, c(R))/q, =_1, \neq_1, \gamma) \rightarrow (Im(\varphi), =_Y, \neq_Y, \beta)$$

where $c(R)$ is the biggest quasi-antiorder relation on X under $R = \alpha \cap Coker(\varphi)$, $q = c(R) \cup c(R)^{-1}$ and γ is an antiorder induced by the quasi-antiorder $c(R)$. If the condition $\alpha \cap \alpha^{-1} = \emptyset$ holds, then the above bijection is isomorphism.

1 Introduction

1.1 Setting. The arguments in this paper conform to Constructive mathematics in the sense of Bishop ([2]). So, our setting is Bishop's constructive mathematics, mathematics developed with Constructive logic (or Intuitionistic logic ([24])) - logic without the Law of Excluded Middle $P \vee \neg P$. We have to note that 'the crazy axiom' $\neg P \Rightarrow (P \Rightarrow Q)$ is included in the Constructive logic. Precisely, in Constructive logic the 'Double Negation Law' $P \Leftrightarrow \neg\neg P$ does not hold, but the following implication $P \Rightarrow \neg\neg P$ holds even in Minimal logic. In Constructive logic 'Weak Law of Excluded Middle' $\neg P \vee \neg\neg P$ does not hold also. It is interesting, in Constructive logic the following deduction principle $A \vee B, \neg A \vdash B$ holds, but this is impossible to prove without 'the crazy axiom'. As Intuitionistic logic is a fragment of Classical logic, our arguments should be valid from a classical point of view.

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1.2 Set with apartness. Let $(X, =, \neq)$ be a set, where " $\neq$ " is a apartness $([2],[4],[8],[12],[14],[23],[24])$. *Apartness* is a binary relation on X which satisfies the following properties:

$$\neg(x \neq x), \quad x \neq y \implies y \neq x, \quad x \neq z \implies (\forall y \in X)(x \neq y \vee y \neq z)$$

for every x, y and z in X . The apartness is *compatible* with the equality in the following sense $(\forall x, y, z \in X)(x = y \wedge y \neq z \implies x \neq z)$. The apartness $\neq$ is *tight* ([23], [24]) if and only if $\neg(x \neq y) \implies x = y$ for any element x and y in X . Let x be an element of X and A subset of X . We write $x \bowtie A$ if $(\forall a \in A)(x \neq a)$, and $A^C = \{x \in X : x \bowtie A\}$.

A relation $q \subseteq X \times X$ is *coequality relation* on X ([10],[12]) if

$$q \subseteq \neq, \quad q^{-1} = q, \quad (\forall x, z \in X)((x, z) \in q \implies (\forall y \in X)((x, y) \in q \vee (y, z) \in q)).$$

The relation $q^C = \{(x, y) \in X \times X : (x, y) \bowtie q\}$ is an equality on X compatible with q , in the following sense

$$(\forall a, b, c \in X)((a, b) \in q^C \wedge (b, c) \in q \implies (a, c) \in q)$$

([18], Theorem 1). We can ([10], [12]) construct factor-sets

$$(X/(q^C, q), =_1, \neq_1) = \{aq^C : a \in X\} \quad \text{and} \quad (X/q, =_1, \neq_1) = \{aq : a \in X\},$$

where

$$\begin{aligned} aq^C =_1 bq^C &\iff (a, b) \bowtie q, & aq^C \neq_1 bq^C &\iff (a, b) \in q, \\ aq =_1 bq &\iff (a, b) \bowtie q, & aq \neq_1 bq &\iff (a, b) \in q. \end{aligned}$$

It is easy to check that $X/(q^C, q) \cong X/q$.

Examples I: (1) The relation $\neg(=)$ is an apartness on the set $\mathbf{Z}$ of integers.

(2) ([8]) The relation q , defined on the set $\mathbf{Q}^N$ by

$$(f, g) \in q \iff (\exists k \in N)(\exists n \in N)(m \geq n \implies |f(m) - g(m)| > k^{-1}),$$

is a coequality relation.

(3) ([8]) A ring R is a local ring if for each $r \in R$, either r or $1 - r$ is a unit. Let M be a module over R . Then the relation q on M , defined by $(x, y) \in q$ if there exists a homomorphism $f : M \longrightarrow R$ such that $f(x - y)$ is a unit, is a coequality relation on M .

(4) ([12]) Let T be a set and $\mathbf{J}$ be a subfamily of $\wp(T)$ such that $\emptyset \in \mathbf{J}$, $A \subseteq B \wedge B \in \mathbf{J} \implies A \in \mathbf{J}$, $A \cap B \in \mathbf{J} \implies A \in \mathbf{J} \vee B \in \mathbf{J}$. If $(X_t)_{t \in T}$ is a family of sets, then the relation q on $\prod X_t$ ($\neq \emptyset$), defined by $(f, g) \in q \iff \{s \in T : f(s) = g(s)\} \in \mathbf{J}$, is a coequality relation on the Cartesian product $\prod X_t$. ♦

1.3 Algebraic structures with apartness. For a function

$$f : (X, =, \neq) \longrightarrow (Y, =, \neq)$$

we say that it is:

- (a) ([8]) *strongly extensional* if and only if $(\forall a, b \in X)(f(a) \neq f(b) \implies a \neq b)$;
- (b) ([7]) an *embedding* if and only if $(\forall a, b \in X)(a \neq b \implies f(a) \neq f(b))$.

In general, all functions in this text are strongly extensional functions. For example, if $\omega : X \times X \longrightarrow X$ is an internal binary operation on X , then must be:

$$(\forall a, b, x, y \in X)(\omega(a, b) \neq \omega(x, y) \implies (a, b) \neq (x, y)).$$

Examples II: Let $(S, =, \neq, \cdot)$ be a semigroup with apartness. Let us note that the internal operation " $\cdot$ " is a strongly extensional function in the following sense:

$$(\forall x, a, b \in S)((ax \neq bx \implies a \neq b) \wedge (xa \neq xb \implies a \neq b)).$$

A subset T of semigroup S is a *right consistent subset* of S ([3]) of S if and only if

$$(\forall x, y \in S)(xy \in T \implies y \in T);$$

a subset T of S is a *left consistent subset* of S ([3]) of S if and only if

$$(\forall x, y \in S)(xy \in T \implies x \in T);$$

a subset T of S is a *consistent subset* of S ([3]) of S if and only if

$$(\forall x, y \in S)(xy \in T \implies x \in T \wedge y \in T).$$

Let q be a coequality relation on a semigroup S such that

$$(\forall a, b, y \in S)((ay, by) \in q \implies (a, b) \in q).$$

Then we say that it is a *left anticongruence* on S . If for q holds

$$(\forall a, b, x \in S)((xa, xb) \in q \implies (a, b) \in q)$$

then q is a *right anticongruence* on S . The coequality relation q on S is an *anticongruence* on S , or relation compatible with semigroup operation on S , if and only if it is a left and right anticongruence.

- (1) ([17]) Let e and f be idempotents of a semigroup S with apartness. Then:
 - (a) the set $X(e) = \{a \in S : ae \neq a\}$ is a strongly extensional right consistent subset of S ;
 - (b) the set $Y(e) = \{b \in S : eb \neq b\}$ is a strongly extensional left consistent subset of S ;
 - (c) the set $P(e) = \{a \in S : e \bowtie Sa\}$ is a strongly extensional left ideal of S ;
 - (d) the set $Q(e) = \{a \in S : e \bowtie aS\}$ is a strongly extensional right ideal of S ;
 - (e) the set $R(e) = \{a \in S : e \bowtie SaS\}$ is a strongly extensional ideal of S such

that $e \bowtie R(e)$;

(f) the set $M(e) = X(e) \cup Y(e) \cup P(e) \cup R(e)$ is a strongly extensional completely prime subset of S such that $e \bowtie M(e)$. Besides, if $e \neq f$, then $M(e) \cup M(f) = S$.

(2) Let $S = \{0\} \times [0, 1]$ ($\subset \mathbf{R} \times \mathbf{R}$, where $\mathbf{R}$ is the set of reals). The multiplication in S is the coordinatewise usual multiplication. Then S is a semigroup with apartness. The set $\{0\} \times [0, >$ is an ideal of S and the set $\{0\} \times [1/2, 1]$ is a consistent subset of S .

(3) The set $S = \{(x, y) \in \mathbf{R} \times \mathbf{R} : x \geq 0 \wedge y \geq 0\}$ with the multiplication on S defined by $(x, y)(a, b) = (xa, xb + y)$ is a semigroup with apartness. The subset $Q = \{(x, y) \in S : x > 0\}$ is a consistent subset of S and filter in S .

(4) Let T be a strongly extensional consistent subset of semigroup S , i.e. let $(\forall x, y \in S)(xy \in T \implies x \in T \wedge y \in T)$. Then relation q on semigroup S , defined by $(a, b) \in q$ if and only if $a \neq b \wedge (a \in T \vee b \in T)$, is a coequality relation on S .

(5) ([18], Theorem 5) Let q be a coequality relation on a semigroup S with apartness. Then the relation $q^+ = \{(x, y) \in S \times S : (\exists a, b \in S^1)((axb, ayb) \in q)\}$ is an anticongruence on S such that $q \subseteq q^+$. If ρ is an anticongruence on S such that $q \subseteq \rho$, then $q^+ \subseteq \rho$. ♦

Examples III: Let $(R, =, \neq, +, 0, \cdot, 1)$ be a commutative ring. A subset Q of R is a *coideal* of R if and only if

$$\begin{aligned} 0 &\bowtie Q, \\ -x \in Q &\implies x \in Q, \\ x + y \in Q &\implies x \in Q \vee y \in Q, \\ xy \in Q &\implies x \in Q \wedge y \in Q. \end{aligned}$$

Coideals of commutative ring with apartness where studied by Ruitenburg 1982 ([23]). After that, coideals (anti-ideals) studied by A.S. Troelstra and D. van Dalen in their monograph [24]. The author proved, in his paper [9], if Q is a coideal of a ring R , then the relation q on R , defined by $(x, y) \in q \iff x - y \in Q$, satisfies the following properties:

(a) q is a coequality relation on R ;

(b) $(\forall x, y, u, v \in R)((x + u, y + v) \in q \implies (x, y) \in q \vee (u, v) \in q)$;

(c) $(\forall x, y, u, v \in R)((xu, yv) \in q \implies (x, y) \in q \vee (u, v) \in q)$.

A relation q on R , which satisfies the properties (a)-(c), is called *anticongruence* on R ([9]) or relation compatible with ring operations. If q is an anticongruence on a ring R , then the set $Q = \{x \in R : (x, 0) \in q\}$ is a coideal of R ([9]). Let J be an ideal of R and if Q is a coideal of R . Wim Ruitenburg, in his dissertation ([23], page 33) first stated a demanded that $J \subseteq \neg Q$. This condition is equivalent with the following condition

$$(\forall x, y \in R)(x \in J \wedge y \in Q \implies x + y \in Q).$$

In this case we say that they are *compatible* ([11]) and we can construct the quotient-ring $R/(J, Q)$. W.Ruitenburg, in his dissertation, first stated the question on the existence an ideal J of R compatible with a given coideal Q and the

question on the existence of a coideal Q of R compatible with a given ideal J . If e is a congruence on R , determined by the ideal J and if q is an anticongruence on R , determined by Q , then J and Q are compatible if and only if

$$(\forall x, y, z \in R)((x, y) \in e \wedge (y, z) \in q \implies (x, z) \in q).$$

In this case we say that e and q are compatible.

(1) Let $R = (R, =, \neq, +, 0, \cdot, 1)$ be a commutative ring with apartness. Then the sets $\emptyset$ and $R = \{a \in R : a \neq 0\}$ are coideals of R . Let a be an element of the ring R . Then the ideal $\text{Ann}(a)$ and the coideal $\text{Cann}(a) = \{x \in R : ax \neq 0\}$ are compatible.

(2) Let m and $i \in \{1, 2, \dots, m-1\}$ be integers. We set $m\mathbf{Z} + i = \{mz + i : z \in \mathbf{Z}\}$. Then the set $\cup\{m\mathbf{Z} + i : i \in \{1, \dots, m-1\}\}$ is a coideal of the ring $\mathbf{Z}$.

(3) Let K be a Richman field and x be an unknown variable under K . Then the set $C = \{f \in K[x] : f(0) \neq 0\}$ is a coideal of the ring $K[x]$.

(4) Let R be a commutative ring. Then the set $B = R^N$ is a ring. For $n \in N$, the set $M = \{f \in B : f(n) \neq 0\}$ is a coideal of B .

(5) Let R be a local ring. Then the set $M = \{a \in R : (\exists x \in R)(ax = 1)\}$ is a coideal of R .

(6) Let S be a coideal of a ring R and let X be a subset of R . Then the set $[S : X] = \{a \in R : (\exists x \in X)(ax \in S)\}$ is a coideal of R .

(7) Let H be a nonempty family of inhabited subsets of T . Then the set $S(H) = \{r \in \prod R_t : (\exists A \in H)(A \cap Z(r) \neq \emptyset)\}$, where $Z(r) = \{t \in T : r(t) \neq 0\}$, is a coideal of the ring $\prod R_t (\neq \emptyset)$. ♦

1.3 Filled product. Let $\alpha \subseteq X \times Y$ and $\beta \subseteq Y \times Z$ be relations. As in [15], we define

$$\beta * \alpha = \{(x, z) \in X \times Z : (\forall y \in Y)((x, y) \in \alpha \vee (y, z) \in \beta)\}.$$

For a relation $R \subseteq X \times X$ we put $^1R = R$, $^nR = R * R * \dots * R$ ($n \geq 2$) and $c(R) = \bigcap_{n \in N} ^nR$. In [14] and [15] this author proved that the relation $c(R)$ is a cotransitive relation under R . This relation is called the *cotransitive fulfillment* of R .

1.4 Goal of this paper. We will briefly recall the constructive definition of linear order and we will use a generalization of J. von Plato ([9]) and M.A. Baroni's ([1]) excess relation for the definition of a partially ordered set. Let X be a nonempty set. A binary relation $<$ (less than) on X is called a *linear order* if the following axioms are satisfied for all elements x and y :

$$\neg(x < y \wedge y < x),$$

$$x < y \implies (\forall z \in S)(x < z \vee z < y).$$

An example is the standard strict order relation $<$ on R , as described in [2], [4], [8] and [9]. For an axiomatic definition of the real number line as a constructive

ordered field, the reader is referred to [2], [4], [9]. A detailed investigation of linear orders in lattices can be found in [9]. The binary relation $\not\leq$ on X is called an excess relation if it satisfies the following axioms:

$$\neg(x \not\leq x), \\ x \not\leq y \implies (\forall z \in S)(x \not\leq z \vee z \not\leq y).$$

Clearly, each linear order is an excess relation. As shown in [9], we obtain an apartness relation $\neq$ and a partial order $\leq$ on X by the following definitions:

$$x \neq y \iff (x \not\leq y \vee y \not\leq x), \\ x \leq y \iff \neg(x \not\leq y).$$

Note that the statement $\neg(x \leq y) \implies x \not\leq y$ does not hold in general.

Let $(X, =, \neq)$ be a set with apartness. A relation $\alpha \subseteq X \times X$ is an *antiorder* relation on X if and only if

$$\alpha \subseteq \neq, \\ (\forall x, y, z \in X)((x, z) \in \alpha \implies (x, y) \in \alpha \vee (y, z) \in \alpha), \\ (\forall x, y \in X)(x \neq y \implies (x, y) \in \alpha \vee (y, x) \in \alpha).$$

A ordered set under an antiorder α is a structure $(X, =, \neq, \alpha)$ where α is an antiorder relation on X . Antiorder relation on a set was first defined by author in paper [14] and [16].

Example IV: Let $S = \{a, b, c, d, e\}$ with multiplication defined by schema

	a	b	c	d	e
a	a	e	c	d	e
b	a	b	c	d	e
c	a	e	c	d	e
d	a	e	c	d	e
e	a	e	c	d	e

The relation $\alpha \subseteq S \times S$, defined by $\alpha = \{(a, b), (a, c), (a, e), (b, a), (b, c), (b, d), (b, e), (c, a), (c, b), (c, d), (d, a), (d, b), (d, c), (d, e), (e, a), (e, b), (e, c), (e, d)\}$ is an antiorder relation on semigroup S . $\blacklozenge$

A relation σ on $(X, =, \neq)$ is a *quasi-antiorder* relation on X if and only if

$$\sigma \subseteq \neq, \\ (\forall x, y, z \in X)((x, z) \in \sigma \implies (x, y) \in \sigma \vee (y, z) \in \sigma).$$

If there exists an antiorder α on the set $(X, =, \neq)$, different from $\neq$, then we have to put a stronger demand in the definition of quasi-antiorder: $\sigma \subseteq \alpha$ instead of $\sigma \subseteq \neq$. A quasi-antiordered set is a structure $(X, =, \neq, \sigma)$ where σ is a quasi-antiorder relation on X . Note that if σ is a quasi-antiorder on X , then σ^{-1} is a quasi-antiorder in X too. Indeed:

- (a) $\sigma \subseteq \neq \implies \sigma^{-1} \subseteq \neq^{-1} = \neq$ (because the relation $\neq$ is symmetric);
 (b) $(x, z) \in \sigma^{-1} \iff (z, x) \in \sigma$
 $\implies (\forall y \in X)((z, y) \in \sigma \vee (y, x) \in \sigma)$
 $\implies (\forall y \in X)((y, z) \in \sigma^{-1} \vee (x, y) \in \sigma^{-1})$
 $\iff (\forall y \in X)((x, y) \in \sigma^{-1} \vee (y, z) \in \sigma^{-1}).$

There is a theory of quasi-order relation in ordered semigroup. See, for example, papers [5] and [6]. In this paper we continue the research parallel relations of antiorder and quasi-antiorder.

Example V: Let $S = \{a, b, c, d, e\}$ with multiplication defined by schema

	a	b	c	d	e
a	b	b	d	d	d
b	b	b	d	d	d
c	d	d	c	d	c
d	d	d	d	d	d
e	d	d	c	d	c

Relation α , defined by $\alpha = \{(a, c), (a, d), (a, e), (b, a), (b, c), (b, d), (b, e), (c, a), (c, b), (c, d), (c, e), (d, a), (d, e), (e, a), (e, b), (e, d)\}$, is an antiorder relation on semigroup S . The relation $\sigma = \{(a, e), (b, e), (c, a), (c, b), (c, d), (c, e), (d, e), (e, a), (e, b), (e, d)\}$ is a quasi-antiorder relation on semigroup S . ♦

The notion of quasi-antiorder relation in set with apartness was introduced for first time in paper [14]. After that, quasi-antiorders are studied by this author in his paper [18], [19], [20] [21], [22]. Sometime, in the definition of antiorder relation on a set $(X, =, \neq)$, we add the condition $\alpha \cap \alpha^{-1} = \emptyset$. In that case, in the definition of quasi-antiorder relation on the ordered set $(X, =, \neq, \alpha)$ under the antiorder α , we must add the following condition $\sigma \cap \sigma^{-1} = \emptyset$. What is different between anti-order relation and excess relation? Clearly, an anti-order relation on set with tight apartness is an excess relation, and, opposite, an excess relation is an anti-order relation.

In this note we proved some kind of isomorphism theorem for ordered sets under antiorders. Let $(X, =_X, \neq_X, \alpha)$ and $(Y, =_Y, \neq_Y, \beta)$ be ordered sets under antiorders, where the apartness $\neq_Y$ is tight. If $\varphi : X \longrightarrow Y$ is reverse isotone function, then there exists a strongly extensional, injective and embedding reverse isotone bijection

$$((X, =_X, \neq_X, \alpha, c(R))/q, =_1, \neq_1, \gamma) \longrightarrow (Im(\varphi), =_Y, \neq_Y, \beta),$$

where $c(R)$ is the biggest quasi-antiorder relation on X under $R = \alpha \cap Coker(\varphi)$, $q = c(R) \cup c(R)^{-1}$ and γ is the antiorder induced by the quasi-antiorder $c(R)$. If the condition $\alpha \cap \alpha^{-1} = \emptyset$ holds, then the above bijection is an isomorphism.

1.5 References. For undefined notions and notations for Constructive mathematics we refer to the books [2], [4], [8], [23] and [24], and to papers [7], [11]-[20]. For classical and constructive semigroup theory we refer to [3], [5], [6] and [7], [17]-[20].

2 Preliminaries

In this section we start with the following explanations:

Remarks A.

(0) A relation q on a set $(X, =, \neq)$ is a coequality relation on X if and only if

$$q \subseteq \neq, q^{-1} = q, q \subseteq q * q.$$

(1) A relation α is an antiorder relation on a set $(X, =, \neq)$ if and only if

$$\alpha \subseteq \neq, \alpha \subseteq \alpha * \alpha, \neq \subseteq \alpha \cup \alpha^{-1}.$$

(2) A relation σ on a ordered set $(X, =, \neq, \alpha)$ under an antiorder α is a quasi-antiorder relation on X iff

$$\sigma \subseteq \alpha, \sigma \subseteq \sigma * \sigma.$$

(3) Sometimes, in the definition of antiorder relation on set $(X, =, \neq)$, we add another condition

$$(\forall x, y \in X)((x, y) \in \alpha \implies \neg((y, x) \in \alpha)),$$

which is equivalent with the condition

$$\alpha \cap \alpha^{-1} = \emptyset.$$

In that case, in the definition of quasi-antiorder relation on the ordered set $(X, =, \neq, \alpha)$ under the antiorder α , we must add the following condition

$$(\forall x, y \in X)((x, y) \in \sigma \implies \neg((y, x) \in \sigma)),$$

i.e. the demand

$$\sigma \cap \sigma^{-1} = \emptyset.$$

Let $(X, =, \neq, \alpha)$, $(Y, =, \neq, \beta)$ be ordered sets under antiorders α and β respectively, $f : X \longrightarrow Y$ a mapping from X into Y . f is called *isotone* if

$$(\forall x, y \in S)((x, y) \in \alpha \implies (f(x), f(y)) \in \beta).$$

f is called *reverse isotone* if and only if

$$(\forall x, y \in S)((f(x), f(y)) \in \beta \implies (x, y) \in \alpha).$$

The mapping f is called an *isomorphism* if it is injective and embedding, onto, isotone and reverse isotone. X and Y are called isomorphic, symbolically $X \cong Y$, if exists an isomorphism between them.

Remarks B.

B.1. Every isotone mapping $f : X \longrightarrow Y$ satisfies the following condition:

- (1) Let $x, y \in X$ and $x \neq_X y$. Then $(x, y) \in \alpha$ or $(y, x) \in \alpha$ by linearity of α and we have $(f(x), f(y)) \in \beta \subseteq \neq_Y$ or $(f(y), f(x)) \in \beta \subseteq \neq_Y$. So, the mapping f is an embedding.
- (2) Let $x, y \in X$ and $f(x) = f(y)$. Then $\neg(f(x) \neq_Y f(y))$, and from this we conclude $\neg((f(x), f(y)) \in \beta)$ and $\neg((f(y), f(x)) \in \beta)$. Hence $\neg((x, y) \in \alpha)$ and $\neg((y, x) \in \alpha)$. Therefore $\neg(x \neq_X y)$. If the apartness $\neq_X$ on set X is tight, then $x = y$. So, in that case when the apartness is tight, the mapping f is an injective.

B.2. Every reverse isotone mapping $f : X \longrightarrow Y$ satisfies the following condition:

- (3) Let $x, y \in X$ such that $f(x) \neq_Y f(y)$. Then $(f(x), f(y)) \in \beta$ or $(f(y), f(x)) \in \beta$ by linearity of β and we have $(x, y) \in \neq_X$ or $(y, x) \in \neq_X$. So, the mapping f is strongly extensional.
- (4) Let $x =_X y$. Then $\neg(x \neq_X y)$, i.e. then $\neg((x, y) \in \alpha \cup \alpha^{-1})$. Suppose that $f(x) \neq_Y f(x)$, i.e. suppose that $(f(x), f(y)) \in \beta \cup \beta^{-1}$. Thus we conclude $(x, y) \in \alpha \cup \alpha^{-1}$ which is impossible. So, our proposition $f(x) \neq_Y f(x)$ is wrong, i.e. must holds $\neg(f(x) \neq_Y f(x))$. If the apartness $\neq_Y$ is tight, then holds $f(x) =_Y f(y)$. So, in this case when the apartness $\neq_X$ is tight, antiorders are compatible with the function f .

Lemma 0: Let σ be a quasi-antiorder relation on an anti-ordered set $(X, =, \neq, \alpha)$. Then $q = \sigma \cup \sigma^{-1}$ is a coequality relation on X such that $(X/q, =_1, \neq_1)$ is an ordered set under the antiorder relation β defined by $(xq, yq) \in \beta \iff (x, y) \in \sigma$.

Proof: Let (uq, vq) be an arbitrary element of β , i.e. let $(u, v) \in \sigma$. Since $\sigma \subseteq q$, we have $uq \neq_1 vq$. Therefore, $\beta \subseteq \neq_1$ (in X/q). Let (xq, zq) and yqX/q , i.e. let (x, z) and yX . Since $(x, y)(y, z)$, we have (xq, yq) or (yq, zq) . Let $(xq, yq) \in \beta$ and $aq, bq \in X/q$, i.e. $(x, y) \in \sigma$ and $a, b \in X$. Let $xq \neq_1 yq$, i.e. let $(x, y) \in q = \sigma \cup \sigma^{-1}$. Since $(x, y) \in \sigma$ or $(y, x) \in \sigma$, we have $(xq, yq) \in \beta$ or $(yq, xq) \in \beta$. So, the relation β is linear. Therefore, the relation β is an antiorder relation on X/q .

Now, suppose that $\sigma \cap \sigma^{-1} = \emptyset$. Then also $\beta \cap \beta^{-1} = \emptyset$. Indeed, let $(xq, yq) \in \beta$, i.e. let $(x, y) \in \sigma$. Then $\neg((y, x) \in \sigma)$, i.e. then $\neg((yq, xq) \in \beta)$. $\square$

Example VI: Let S, α and σ as in the example II. Then the relation $q = \sigma \cup \sigma^{-1} = \{(a, e), (b, e), (c, a), (c, b), (c, d), (c, e), (e, a), (e, b), (e, d), (e, a), (e, b), (a, c), (b, c), (d, c), (e, c), (a, e), (b, e), (d, e)\}$ is an anticongruence on S . Then $aq = \{c, e\}$, $bq = \{c, e\}$, $cq = \{a, b, d, e\}$, $dq = \{c, e\}$, $eq = \{a, b, c, d\}$ and $S/q = \{\{c, e\}, \{a, b, d, e\}, \{a, b, c, d\}\}$. So, the relation β is defined in the following way: $\beta =$

$\{(aq, eq), (bq, eq), (cq, aq), (cq, bq), (cq, dq), (cq, eq), (eq, aq), (eq, bq), (eq, dq)\}.$

◆

Corollary 0.1: *The mapping $\pi : X \longrightarrow X/q$ is a reverse isotone surjective function.*

Lemma 1: *If $\{\sigma_k\}_{k \in J}$ is a family of quasi-antiorders on a set $(X, =, \neq)$ relatively to a certain antiorder α , then $\bigcup_{k \in J} \sigma_k$ is a quasi-antiorder in X .*

Proof: Let (x, z) be an arbitrary elements of $X \times X$ such that $(x, z) \in \bigcup_{k \in J} \sigma_k$. Then there exists k in J such that $(x, z) \in \sigma_k$. Hence for every $y \in X$ we have $(x, y) \in \sigma_k \vee (y, z) \in \sigma_k$. So, $(x, y) \in \bigcup_{k \in J} \sigma_k \vee (y, z) \in \bigcup_{k \in J} \sigma_k$. At the other side, for every k in J holds $\sigma_k \subseteq \alpha$. From this we conclude $\bigcup_{k \in J} \sigma_k \subseteq \alpha$. $\square$

3 The main results

First, we show a construction of maximal quasi-antiorder under a given relation:

Theorem 3: *Let $R (\subseteq \neq)$ be a relation on a set $(X, =, \neq)$. Then for an inhabited family of quasi-antiorders under R there exists the biggest quasi-antiorder relation under R . That relation is exactly the relation $c(R)$.*

Proof: By Lemma 1, there exists the biggest quasi-antiorder relation on X under R . Let $\mathbf{A}_R$ be the inhabited family of all quasi-antiorder relation on X under R . With (R) we denote the biggest quasi-antiorder relation $\cup \mathbf{A}_R$ on X under R . The fulfillment $c(R) = \bigcap_{n \in \mathbb{N}} {}^n R$ of the relation R is a cotransitive relation on set X under R . Therefore, $c(R) \subseteq (R)$ holds.

We need to show that $(R) \subseteq c(R)$. Let s be a quasi-antiorder relation in X under R . First, we have $s \subseteq R = {}^1 R$. Let $(x, z) \in s$. Then from $(\forall y \in X)((x, y) \in s \vee (y, z) \in s)$ we conclude that for every y in X holds $(x, y) \in R \vee (y, z) \in R$, i.e. holds $(x, z) \in R * R = {}^2 R$. So, $s \subseteq {}^2 R$. Now, we will suppose that $s \subseteq {}^n R$ and let $(x, z) \in s$. Then from $(\forall y \in X)((x, y) \in s \vee (y, z) \in s)$ implies that $(x, y) \in R \vee (y, z) \in {}^n R$ holds for every $y \in X$. Therefore, $(x, z) \in {}^{n+1} R$. So, we have $s \subseteq {}^{n+1} R$. Thus, by induction, we have $s \subseteq \bigcap {}^n R$. Remember that s is an arbitrary quasi-antiorder on X under R . Hence, we proved that $(R) = \cup \mathbf{A}_R \subseteq c(R)$. $\square$

Corollary 3.1: *Let $(X, =, \neq, \alpha)$ be an ordered set under an antiorder α . Then the family $\mathbf{A} = \{\tau : \tau \text{ is a quasi-antiorder on } X \text{ under } \alpha\}$ is a complete lattice.*

Example VII ([18]): Let a and b be elements of semigroup S . Then ([18], Theorem 6) the set - $C_{(a)} = \{x \in S : x \bowtie SaS\}$ is a consistent subset of S such that :

- $a \bowtie C_{(a)}$;
- $C_{(a)} \neq \emptyset \implies 1 \in C_{(a)}$;

- Let a be an invertible element of S . Then $C_{(a)} = \emptyset$;
- $(\forall x, y \in S)(C_{(a)} \subseteq C_{(xay)})$;
- $C_{(a)} \cup C_{(b)} \subseteq C_{(ab)}$.

Let a be an arbitrary element of a semigroup S with apartness. The consistent subset $C_{(a)}$ is called a *principal* consistent subset of S generated by a . We introduce relation f , defined by $(a, b) \in f \iff b \in C_{(a)}$ and in the next assertion we will give some description of the relation f : The relation f has the following properties ([17], Theorem 7)

- f is a consistent relation ;
- $(a, b) \in f \implies (\forall x, y \in S)((xay, b) \in f)$;
- $(a, b) \in f \implies (\forall n \in N)((a^n, b) \in f)$;
- $(\forall x, y \in S)((a, xby) \in f \implies (a, b) \in f)$;
- $(\forall x, y \in S)\neg((a, xay) \in f)$.

We can construct the cotransitive relation $c(f) = \bigcap_{n \in N} {}^n f$ as cotransitive fulfillment of the relation f ([7],[14],[18]). As corollary of theses assertions we have the following results: The relation $c(f)$ satisfies the following properties:

- $c(f)$ is a consistent relation on S ;
- $c(f)$ is a cotransitive relation ;
- $(\forall x, y \in S)((a, xay) \bowtie c(f))$;
- $(\forall n \in N)((a, an) \bowtie c(f))$;
- $(\forall x, y \in S)((a, b) \in c(f) \implies (xay, b) \in c(f))$;
- $(\forall n \in N)((a, b) \in c(f) \implies (a^n, b) \in c(f))$;
- $(\forall x, y \in S)((a, xby) \in c(f) \implies (a, b) \in c(f))$.

For an element a of a semigroup S and for $n \in N$ we introduce the following notations

$$A_n(a) = \{x \in S : (a, x) \in {}^n f\}, \quad A(a) = \{x \in S : (a, x) \in c(f)\}$$

By the following results we will present some basic characteristics of these sets. Let a and b be elements of a semigroup S . Then:

- $A_1(a) = C_{(a)}$;
- $A_{n+1}(a) \subseteq A_n(a)$;
- $A_{n+1}(a) = \{x \in S : S = A_n(a) \cup B_1(x)\}$ where $B_1(x) = \{u \in S : (u, x) \in f\}$;
- $A(a) = \bigcap_{n \in N} A_n(a)$;
- $a \bowtie A(a)$;
- $A(a) \cup A(b) \subseteq A(ab)$;
- The set $A(a)$ is the maximal strongly extensional consistent subset of S such that $a \bowtie A(a)$. ♦

Example VIII: Let $S = \{a, b, c, d\}$ be a ordered semigroup with the Cayley table and antiorder shown below:

	a	b	c	d
a	b	b	c	c
b	b	b	c	c
c	c	c	c	c
d	c	c	c	c

$\alpha = \{(a, b), (a, c), (a, d), (b, a), (c, a), (c, b), (d, a), (d, b), (d, c)\}$. For relation α holds $\alpha \cap \alpha^{-1} \neq \emptyset$. If we put $\beta = \{(a, b), (a, c), (a, d), (c, b), (d, b), (d, c)\}$, then $\beta \subset \alpha$ and $\beta \cap \beta^{-1} = \emptyset$. ♦

Let $f : X \longrightarrow Y$ be a strongly extensional function. It is easy to verify that sets

$$\begin{aligned} \text{Ker}(f) &= \{(a, b) \in X \times X : f(a) = f(b)\}, \\ (q =) \text{Coker}(f) &= \{(a, b) \in X \times X : f(a) \neq f(b)\} \end{aligned}$$

are compatible equality and coequality relations on X and we can construct the factor-set X/q .

The following theorem is the main result in this paper:

Theorem 4: *Let $(X, =_X, \neq_X, \alpha)$ and $(Y, =_Y, \neq_Y, \beta)$ be ordered sets under antiorders, where the apartness $\neq_Y$ is tight. If $\varphi : X \longrightarrow Y$ is reverse isotone strongly extensional function, then there exists a strongly extensional and embedding reverse isotone bijection*

$$((X, =_X, \neq_X, \alpha, c(R))/q, =_1, \neq_1, \gamma) \longrightarrow (Im(\varphi), =_Y, \neq_Y, \beta)$$

where $c(R)$ is the biggest quasi-antiorder relation on X under $R = \alpha \cap \text{Coker}(\varphi)$, $q = c(R) \cup c(R)^{-1}$ and γ is the antiorder induced by the quasi-antiorder $c(R)$. If the condition $\alpha \cap \alpha^{-1} = \emptyset$ holds, then there exists the isomorphism

$$(X, =_X, \neq_X, \alpha, c(R))/q, =_1, \neq_1, \gamma \cong (Im(\varphi), =_Y, \neq_Y, \beta).$$

Proof:

(1) Let $(X, =_X, \neq_X, \alpha)$ and $(Y, =_Y, \neq_Y, \beta)$ be ordered sets under antiorders α and β respectively, and $\varphi : X \longrightarrow Y$ a strongly extensional mapping. Then the relation $\varphi^{-1}(\beta) = \{(a, b) \in X \times X : (\varphi(a), \varphi(b)) \in \beta\}$ is a quasi-antiorder on X , the relation $\text{Coker} = \{(a, b) \in X \times X : \varphi(a) \neq_Y \varphi(b)\}$ is coequality relation on X compatible with equality relation $\text{Ker}\varphi = \varphi^{-1} \circ \varphi$, and $\text{Coker}\varphi \supseteq \varphi^{-1}(\beta) \cup (\varphi^{-1}(\beta))^{-1}$ holds. Also, since the relation β is linear we have $\text{Coker}\varphi = \varphi^{-1}(\beta) \cup (\varphi^{-1}(\beta))^{-1}$. Indeed, $\alpha \subseteq \neq$ and $\varphi^{-1}(\beta) \subseteq \alpha$.

Since the relation β is linear, we have

$$\begin{aligned} (a, b) \in \text{Coker}\varphi &\iff \varphi(a) \neq_Y \varphi(b) \\ &\implies (\varphi(a), \varphi(b)) \in \beta \vee (\varphi(b), \varphi(a)) \in \beta \\ &\iff (a, b) \in \varphi^{-1}(\beta) \vee (b, a) \in \varphi^{-1}(\beta). \end{aligned}$$

At the other side, if $(a, b) \in \varphi^{-1}(\beta)$ or $(b, a) \in \varphi^{-1}(\beta)$, then $(\varphi(a), \varphi(b)) \in$

$\beta \subseteq \neq_Y$ or $(\varphi(b), \varphi(a)) \in \beta \subseteq \neq_Y$. Therefore, $Coker\varphi = \varphi^{-1}(\beta)(\varphi^{-1}(\beta))^{-1}$.

(2) It is easily to conclude that $Coker(\varphi) \subseteq \alpha \cup \alpha^{-1}$.

(3) The family $\mathbf{A}_R$ of quasi-antiorder relations on X under relation $R = \alpha \cap Coker(\varphi)$ is not empty, because $\varphi^{-1}(\beta) \subseteq R$. Then, by Theorem 3, there exists the biggest quasi-antiorder relation $c(R)$ under R . Put $q = c(R) \cup c(R)^{-1}$. We can construct, according Lemma 0, the ordered factor-set $((X, =_X, \neq_X, c(R))/q, =_1, \neq_1)$ under antiorder relation γ on X/q , defined by $(aq, bq) \in \gamma$ if and only if $(a, b) \in c(R)$.

(4) We wish to show that $Coker(\varphi) = q = c(R) \cup (c(R))^{-1}$. The first, by definition of $c(R)$, $c(R)$ is the biggest quasi-antiorder relation under R . So, we have $c(R) \subseteq Coker(\varphi)$ and $(c(R))^{-1} \subseteq (Coker(\varphi))^{-1} = Coker(\varphi)$ because the relation $Coker(\varphi)$ is symmetric. Therefore, holds $c(R) \cup (c(R))^{-1} \subseteq Coker(\varphi)$. The second, the relation $\varphi^{-1}(\beta)$ is a quasi-antiorder under $R = \alpha \cap Coker(\varphi)$. So, it must be $\varphi^{-1}(\beta) \subseteq c(R)$ because the relation $c(R)$ is the biggest under R . Thus, it must be $(\varphi^{-1}(\beta))^{-1} \subseteq (c(R))^{-1}$. Therefore, it must be $Coker(\varphi) = \varphi^{-1}(\beta) \cup (\varphi^{-1}(\beta))^{-1} \subseteq c(R) \cup (c(R))^{-1}$.

If $\alpha \cap \alpha^{-1} = \emptyset$ holds, then easy to verify that $c(R) \cap (c(R))^{-1} = \emptyset$ holds too.

(5) By Lemma 0, the set $((X, =_X, \neq_X, \alpha)/q, =_1, \neq_1)$ is ordered set under the antiorder γ on X/q defined by

$$(aq, bq) \in \gamma \iff (a, b) \in c(R).$$

If $\alpha \cap \alpha^{-1} = \emptyset$, then $c(R) \cap (c(R))^{-1} = \emptyset$, because $c(R) \cup (c(R))^{-1} \subseteq \alpha \cup \alpha^{-1}$. It remains to construct mapping $\phi : X/q \longrightarrow Im(\varphi) (\subseteq Y)$. Define $\phi(aq) = \varphi(a)$ for any a in X .

(a) This mapping is well defined because if $aq =_1 bq$, i.e. if $(a, b) \bowtie q = c(R) \cup (c(R))^{-1} = Coker(\varphi)$, then $\neg(\varphi(a) \neq_Y \varphi(b))$ holds. Since the apartness $\neq_Y$ is tight, it implies that $\varphi(a) =_Y \varphi(b)$, i.e. $\phi(aq) =_Y \phi(bq)$.

(b) Suppose that $\phi(aq) \neq_Y \phi(bq)$, i.e. suppose that $\varphi(a) \neq_Y \varphi(b)$, i.e. suppose that $(a, b) \in Coker(\varphi)$. Then $aq \neq_1 bq$. Therefore, the mapping is strongly extensional function from set X/q into Y .

(c) If $y \in Im(\varphi)$, then for some $x \in X$, $\phi(xq) =_Y \varphi(x) =_Y y$. Thus, the mapping $\phi : X/q \longrightarrow Im(\varphi)$ is a strongly extensional and surjective function.

(d) If $\phi(aq) =_Y \phi(bq)$, then $\varphi(a) =_Y \varphi(b)$. Let (u, v) be an arbitrary element of $Coker(\varphi)$. Then from $\varphi(u) \neq_Y \varphi(v)$ follows

$$\varphi(u) \neq_Y \varphi(a) \vee \varphi(a) \neq_Y \varphi(b) \vee \varphi(b) \neq_Y \varphi(v).$$

Since $\varphi(a) \neq_Y \varphi(b)$ is impossible, we conclude that above disjunction follows

$$\varphi(u) \neq_Y \varphi(a) \vee \varphi(b) \neq_Y \varphi(v)$$

and $u \neq_X a$ or $b \neq_X v$. So, $(u, v) \neq_{X \times Y} (a, b)$. This means $(a, b) \bowtie Coker(\varphi)$. Therefore $aq =_1 bq$. Hence, the mapping ϕ is an injective function.

(e) Now, let be $aq \neq_1 bq$. Then $(a, b) \in Coker(\varphi)$, i.e. then $\varphi(a) \neq_Y \varphi(b)$.

Therefore, in this case, we have $\phi(aq) \neq_Y \phi(bq)$. So, the function ϕ is an embedding.

(f) The first, we wish to prove that the function ϕ is reverse isotone bijection. If $(\phi(aq), \phi(bq)) \in \beta$, i.e. if $(\varphi(a), \varphi(b)) \in \beta (\subseteq \neq_Y)$, then $(a, b) \in \varphi^{-1}(\beta) \subseteq c(R)$ by the second part of the point (3) of this proof. Therefore, $(aq, bq) \in \gamma$. So, the bijection is reverse isotone.

The second, we wish to prove that the function ϕ is isotone bijection. Let $(aq, bq) \in \gamma (\subseteq \neq_1)$, i.e. let $(a, b) \in c(R) (\subseteq \alpha)$. Since the function ϕ is an embedding, then $\phi(aq) \neq_Y \phi(bq)$. So, must be $(\phi(aq), \phi(bq)) \in \beta$ or $(\phi(bq), \phi(aq)) \in \beta$. Suppose that $(\phi(bq), \phi(aq)) \in \beta$, i.e. suppose that $(\varphi(b), \varphi(a)) \in \beta$ holds. Thus we conclude that $(b, a) \in \alpha$ because the function φ is reverse isotone mapping. If the condition $\alpha \cap \alpha^{-1} = \emptyset$ holds, then the case $(\phi(bq), \phi(aq)) \in \beta$ is impossible. Now, we have to have $(\phi(aq), \phi(bq)) \in \beta$. So, in the case that the condition $\alpha \cap \alpha^{-1} = \emptyset$ holds, the mapping ϕ is isotone.

At end of this conclusion we have that there exists strongly extensional and embedding reverse isotone bijection from $((X, =_X, \neq_X, \alpha, c(R))/q, =_1, \neq_1)$ onto $(Im(\varphi), =_Y, \neq_Y, \gamma)$. If the condition $\alpha \cap \alpha^{-1} = \emptyset$ holds, then there exists the isomorphism $((X, =_X, \neq_X, \alpha, c(R))/q, =_1, \neq_1, \gamma) \cong (Im(\varphi), =_Y, \neq_Y, \gamma)$. $\square$

Note. Let $(X, =, \neq, \alpha)$, $(Y, =, \neq, \beta)$ be ordered sets under antiorders α and β respective, and let $\varphi : X \rightarrow Y$ be a strongly extensional mapping from X into Y . Then, by point (1) in the proof of the Theorem 4, the relation induced there $\varphi^{-1}(\beta)$ is quasi-antiorder relation on X . Then:

- (i) φ is isotone if $\alpha \subseteq \varphi^{-1}(\beta)$;
- (ii) φ is reverse isotone if and only if $\varphi^{-1}(\beta) \subseteq \alpha$.

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